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Verification of the Benjamin–Lighthill conjecture about steady water waves

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The exact problem of steady periodic waves (Stokes waves) on the surface of an ideal liquid above a horizontal bottom is reconsidered in order to confirm a general property conjectured by Benjamin & Lighthill (1954). Specifically, in terms of parameters r and s proportional respectively to the total-head and flow-force constants for steady flows, such waves are proved to realize points (r, s) inside the region of the (r, s)-plane that is bounded by the cusped curve representing all possible uniform streams. A corresponding attribute of steady periodic waves on the surface of an infinitely deep ideal liquid will also be demonstrated. The concluding discussion refers to steady water waves that are not periodic Stokes waves, and comments with reference to Appendix B on the significance of the flow-force invariant s in Hamiltonian representations of the steady-wave problem.

1. Introduction

It was first shown by Benjamin & Lighthill (1954: to be cited hereafter as BL) how the class of steady water waves in a horizontal open channel of uniform rectangular cross-section is representable by three constants of the wave motion. These parameters are Q the volume flux (per unit span), R the total head (Bernoulli constant), and S the flow force (horizontal momentum flux plus pressure force per unit span, divided by the density of water). Although over 40 years the BL characterization of steady water waves has become widely accepted as a cornerstone of the subject, a pivotal conjecture about the interdependent possible ranges of Q, R and S still remains unproven. The issue will at last be settled in this paper.

To preface the present analysis, the main points of BL's original treatment deserve to be recalled. For steady long waves of small amplitude, they showed that the water depth $H(X)$, a function of the horizontal coordinate X along the channel, is approximated by solutions of the equation

$$\frac{1}{3}Q^2\left(\frac{\mathrm{d}H}{\mathrm{d}X}\right)^2 + gH^3 - 2RH^2 + 2SH - Q^2 = 0 \tag{1.1}$$

(BL, equation (20)). In terms of dimensionless variables $h = H/H_c$ and $x = X/H_c$, where $H_c = (Q^2/g)^{1/3}$ is the depth of the uniform stream that is *critical* for a given Q (i.e. velocity $Q/H_c = (gH_c)^{1/2}$), this equation becomes

$$\frac{1}{3}\left(\frac{\mathrm{d}h}{\mathrm{d}x}\right)^2 + C(h) = 0 \tag{1.2}$$

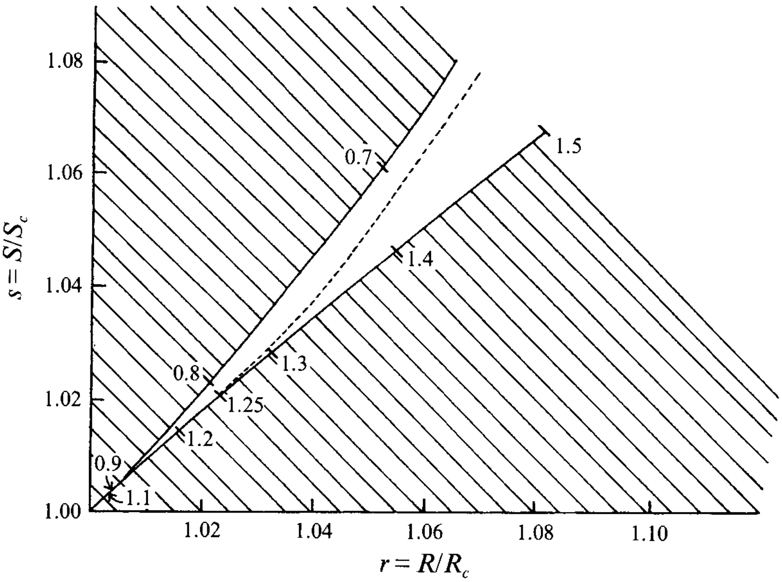

FIGURE 1. Possible values of (r, s) according to Benjamin & Lighthill (1954). The cusped curve is the locus of uniform streams, parameterized by $F = h^{-3/2}$.

with

$$C(h) = h^3 - 3rh^2 + 3sh - 1 \tag{1.3}$$

(BL, equation (23)), in which the two parameters are defined by

$$r = \frac{R}{R_c} = \frac{2R}{3gH_c}, \qquad s = \frac{S}{S_c} = \frac{2S}{3gH_c^2}. \tag{1.4}$$

Note that uniform streams have

$$r = \tfrac{1}{3}(2h + h^{-2}), \qquad s = \tfrac{1}{3}(h^2 + 2h^{-1}), \tag{1.5}$$

so that $r \geqslant 1$ and $s \geqslant 1$ with equality only in the case $h = 1$ of a critical stream. Parameterized by $h > 0$, the possible pairs of values (r, s) for uniform streams lie on the cusped curve shown in figure 1 (BL, figure 2 which has often been reproduced). On the upper branch of the cusp ($h > 1$) the respective uniform streams are subcritical (i.e. Froude number $F = (Q^2/gH^3)^{1/2} = h^{-3/2} < 1$), and on the lower branch they are supercritical. For each point (r, s) on either branch, the cubic $C(h)$ has a double root: i.e. $C(h) = 0$ and $C'(h) = 0$.

Justified as a rational approximation for long waves of small amplitude, equation (1.2) recovers the class of solitary and periodic (cnoidal) waves first identified collectively by Korteweg & de Vries (1895). The approximation requires both $r - 1 > 0$ and $s - 1 > 0$ to be small. Let $h_1 > 1$ and $h_2 < 1$ denote the two positive values of h satisfying the first of (1.5) for a given $r > 1$. The respective values of s satisfy $s_1 > s_2$. If $s = s_2$, $C(h)$ is negative for h between its double root h_2 and its greater root $\hat{h} = h_2^{-2} > 1$. Then (1.2) has a non-trivial solution describing the solitary wave that can arise on the supercritical stream, having the same (rescaled) total head r and flow force s as the undisturbed stream. The height of the solitary wave at its crest is $\hat{h}$. If $s_2 < s < s_1$, the discriminant of the cubic C is positive, and $C(h)$ is negative for h between distinct roots $\check{h}$ and $\hat{h}$, where $h_2 < \check{h} < h_1 < \hat{h}$ and $\hat{h}$ is now smaller than

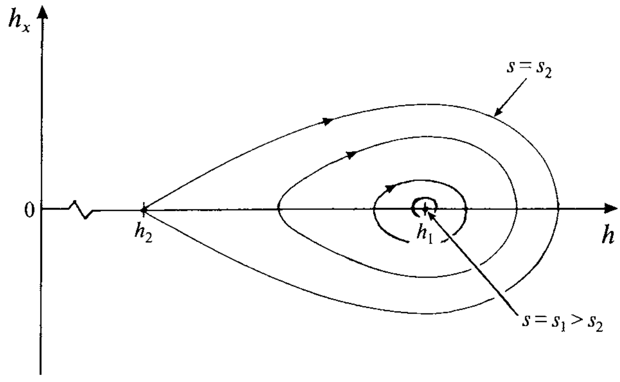

FIGURE 2. Solutions of (1.2) in phase-plane (h, h_x), parameterized by $s \in [s_2, s_1]$ for a fixed $r > 1$.

in the case $s = s_2$ (cf. BL, figure 1). Then (1.2) has a periodic, cnoidal-wave solution with $h = \check{h}$ at the troughs of the wave and $h = \hat{h}$ at the crests. The wave amplitude $\hat{h} - \check{h}$ decreases continuously from its largest value $h_2^{-2} - h_2$ to zero as s increases from s_2 to s_1.

This range of possibilities, which is very well known, can best be illustrated by trajectories of solutions in the phase-plane (h, h_x). Figure 2 shows the solutions of (1.2) parameterized by $s \in [s_2, s_1]$ for fixed $r > 1$. When $s = s_2$ the solution is a homoclinic orbit starting and ending at $[h_2, 0]$; and when $s_2 < s < s_1$ the solutions are periodic orbits around the centre $[h_1, 0]$. When $s \notin [s_1, s_2]$, or more generally whenever (r, s) lies outside the cusped region shown in figure 1, (1.2) has no non-trivial solution bounded on $\boldsymbol{R}$, nor any solution representing a uniform stream, which requires $C'(h) = 0$ as well as $C(h) = 0$.

A similar range of possibilities for fixed s may evidently be parameterized by $r \in [r_2, r_1]$, where r_1 and $r_2 < r_1$, respective to supercritical and subcritical uniform streams, are the two values of r according to (1.5) with $s > 1$ given. Physical interpretations of this and the preceding case were pointed out by BL. Namely, a bore (hydraulic jump) can be modelled as a supercritical–subcritical transition between uniform streams with the same s, the difference $r_1 - r_2 > 0$ being accountable to energy loss in the bore (cf. Lamb 1932, p. 280). A weak, undular bore is explainable by an energy loss less than the maximum $r_1 - r_2$ at its front. Again, a subcritical–supercritical transition between uniform streams without energy loss, such as may be brought about by a sluice-gate or other obstacle spanning the flow, is represented by a vertical jump from the upper to the lower branches of the cusped curve in figure 1, the difference $s_1 - s_2 > 0$ being accountable to the horizontal force holding the obstacle in place. Wave-trains generated in the wake of an obstacle inserted into a subcritical stream are explainable by a reduction in s less than the maximum $s_1 - s_2$, which flow-force reduction is commonly termed wave resistance (cf. Lamb 1932, §249; Benjamin 1956).

Now, BL conjectured that *all* steady two-dimensional irrotational wave motions in an open channel, irrespective of amplitude or wavelength, are represented by points (r, s) on or inside the cusped region in figure 1. Thus the unifying property that is unequivocal in relation to the original model for small-amplitude long waves was guessed to hold generally over the whole class of steady gravity waves. It has been proved by Keady & Pritchard (1974), Amick & Toland (1981*a*) and McLeod (1984) that solitary waves can occur only on supercritical streams (i.e. with (r, s) on the lower branch of the cusp), and so the conjecture can be delimited to periodic waves. Attention also needs to be directed, however, to the examples of steady waves with subharmonic dependence that have been investigated by Chen & Saffman (1980), Vanden-Broeck (1983), Zufiria (1987*a*) and others.

Support for the BL conjecture was provided by De (1955), who developed approximations for Stokes waves to fifth order in wave amplitude. All values of r and s calculated by him are commensurate with the conjecture, the general truth of which has been presumed by many subsequent writers on the subject, notwithstanding that details of De's contribution have been heavily criticized (Chappelear 1961; Fenton 1985; Dixon 1989). Although not presented explicitly in terms of r and s, numerical calculations of R and S were reported by Cokelet (1977) which too are confirmatory. The conjecture was re-emphasized by Keady & Norbury (1975, p. 669, Conjecture 2), who deduced various rigorous bounds for the properties of water waves according to ideal-fluid theory. Several of these bounds will be recovered in what follows, accordingly their paper will hereafter be cited as KN for convenience, and to a large extent their notation will be copied (although r and s here and in BL are R and S in KN). It is relevant to cite also Keady & Norbury (1978*a*), where corresponding bounds were cleverly established for surface waves in ideal liquids with prescribed distributions of vorticity. Until now, however, the question whether the BL conjecture is generally true has remained open.

The main purpose of the present paper is to verify the conjectured attribute of all steady irrotational wave motions in water of finite depth, removing a long-standing uncertainty of the subject. In §4 the corresponding general property of steady surface waves on water of infinite depth will also be proven.

2. The problem for periodic water waves

Take axes (x, y) as shown in figure 3, with origin at the horizontal bottom of the channel below a trough of a steady periodic wave-train. Let λ denote the wavelength. Here the scheme of dimensionless variables introduced in §1 is used, according to which the volume flux is unity. The free surface is described by

$$y = h(x),$$

where h is a periodic real function ranging between minimum and maximum values

$$h(0) = \check{h} > 0 \quad \text{and} \quad h(\tfrac{1}{2}\lambda) = \hat{h} > \check{h}.$$

Treating the water as an incompressible ideal liquid of unit density, we suppose that the flow is steady, two-dimensional and irrotational. The complex potential for it is written $\chi = \phi + \mathrm{i}\psi$, and the complex velocity is $w = u - \mathrm{i}v = \mathrm{d}\chi/\mathrm{d}z$, where $z = x + \mathrm{i}y$. Thus ϕ, ψ, u and v are bounded harmonic functions of (x, y) in the strip

$$D = \boldsymbol{R} \times (0, h(x)).$$

The boundary conditions on the stream function ψ, which satisfies $\Delta\psi = 0$ in D, include

$$\psi(x, 0) = 0 \quad \text{for all} \quad x \in \boldsymbol{R} \tag{2.1}$$

and

$$\psi(x, h(x)) = 1 \quad \text{for all} \quad x \in \boldsymbol{R}. \tag{2.2}$$

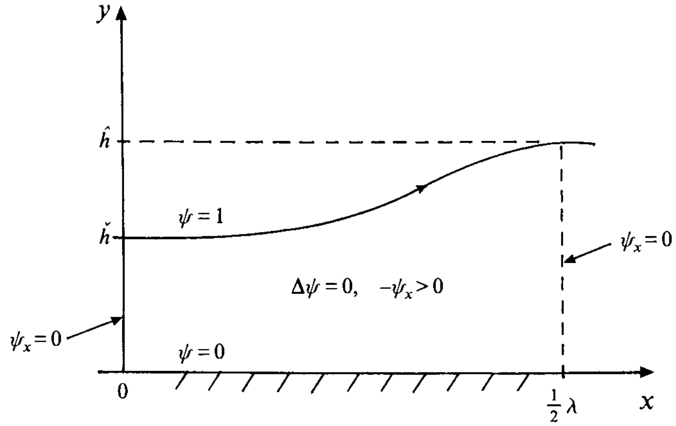

FIGURE 3. Half a wavelength of steady wave-train in the (x, y)-plane.

In addition, Bernoulli's equation at the free surface, assumed to be free of surface tension, is

$$h + \tfrac{1}{2}q^2 = \tfrac{3}{2}r = \text{const.}, \tag{2.3}$$

where

$$q = |w(x, h(x))| = |\nabla\psi| \qquad \text{at} \qquad y = h(x).$$

Periodic-wave solutions of this problem, such that $u = \psi_y > 0$ everywhere in D and $v = -\psi_x > 0$ for $0 < x < \frac{1}{2}\lambda$, $0 < y \leqslant h(x)$, have been proved to exist (Krasovskii 1961; Keady & Norbury 1978*b*). The best known existence theories depend on a reformulation of the problem in terms of $y(\phi, \psi)$, a harmonic function of (ϕ, ψ) in the uniform strip $\boldsymbol{R} \times (0, 1)$ rather than D for (x, y); but this alternative formulation has no advantage for present purposes.

The flow-force invariant for steady waves can be deduced from the expression for horizontal momentum flux plus pressure force, namely

$$\tfrac{3}{2}s = \int_0^h (u^2 + p)\,\mathrm{d}y,$$

in which p is pressure. Eliminating p by means of Bernoulli's equation $p + y + \frac{1}{2}(u^2 + v^2) = \frac{3}{2}r$, one obtains directly

$$\tfrac{3}{2}s = \tfrac{3}{2}rh - \tfrac{1}{2}h^2 + \int_0^h \tfrac{1}{2}(\psi_y^2 - \psi_x^2)\,\mathrm{d}y \tag{2.4}$$

(cf. Benjamin 1984, §6.4, equation (6.8)). In view of (2.3), (2.1) and (2.2), the latter of which implies that $\psi_x + \psi_y h_x = 0$ at $y = h(x)$, it is easy to confirm from (2.4) that $\mathrm{d}s/\mathrm{d}x = 0$. Note that equation (1.2) originally found by BL is obtainable directly from (2.4) by substituting $\psi = y/h(x)$, which is justifiable as a first approximation for long waves of small amplitude (cf. Benjamin 1984, §6.5).

The following interdependent properties of steady periodic waves have been checked by many authors (e.g. Keady & Norbury 1978*b*; Amick & Toland 1981*b*) and will be shown in Appendix A to be inferable from a basic, much more meagre set of properties.

(i) The stream function $\psi(x, y)$, $u = \psi_y(x, y)$ and $h(x)$ are even functions of x and of $x - \frac{1}{2}\lambda$, so that $v = -\psi_x(x, y)$ is an odd function of x and of $x - \frac{1}{2}\lambda$.

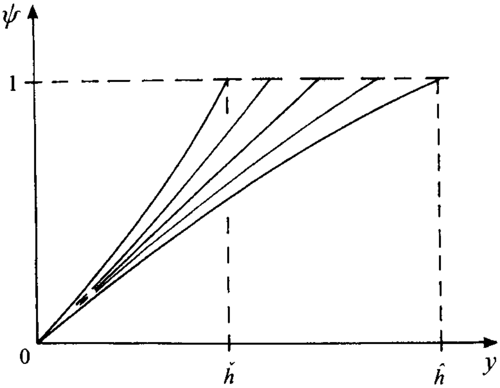

FIGURE 4. Graphs of ψ versus y for various values of x over half a wavelength of steady wave-train.

(ii) The maximum and minimum values of $|\nabla\psi|$ occur on the free boundary $y = h(x)$ of D, the maximum $\check{q}$ at the wave trough where $y = h(0) = \check{h}$, and the minimum $\hat{q}$ at the crests where $y = h(\frac{1}{2}\lambda) = \hat{h}$.

(iii) At the troughs, $u_y = v_x = \psi_{yy} > 0$ for $0 < y \leqslant \check{h}$. At the crests, $u_y < 0$ for $0 < y \leqslant \hat{h}$.

(iv) Above the bottom $\psi = 0$, each of the streamlines $\psi =$ const. > 0 including the free surface $\psi = 1$ rises monotonically from its lowest point at $x = 0$ to its highest point at $x = \frac{1}{2}\lambda$. That is, if the streamlines are described by equations $y = \zeta(x;\psi)$ parameterized by $\psi \in [0,1]$ (e.g. $\zeta(x;1) = h(x)$), then ζ is a monotonic increasing function of x in $[0, \frac{1}{2}\lambda]$ for each $\psi \in (0,1]$. Because $\psi_x + \psi_y\zeta_x = 0$ on $y = \zeta$, this property is subsumed by $u = \psi_y > 0$ everywhere in D (see (v) below) and $v = -\psi_x > 0$ for $0 < x < \frac{1}{2}\lambda$, $0 < y \leqslant h(x)$.

(v) Along the bottom $\psi = 0$, u is a monotonic decreasing function of x in $[0, \frac{1}{2}\lambda]$; and $u > 0$ everywhere in D, except in the case of waves of extreme form for which at the crests $\hat{q} = 0$ and so $\hat{h} = \frac{3}{2}r$.

Figure 4 illustrates the forms of ψ as a function of y at successive stations x between $x = 0$ and $x = \frac{1}{2}\lambda$. In the light of properties (iii) - (v), it is evident that the curves $\psi = \psi(0, y)$ and $\psi = \psi(\frac{1}{2}\lambda, y)$ intersect only at the origin, and the region bounded by these curves and the straight line $\psi = 1$ is covered injectively by the field $\mathscr{F}$ of curves $\psi = \psi(x, y)$ in the (y, ψ)-plane with $x \in [0, \frac{1}{2}\lambda]$. Hence any straight line $\psi = \alpha y$, where $u(\frac{1}{2}\lambda, 0) < \alpha < u(0, 0)$, crosses the interior of the region in question and is covered by $\mathscr{F}$ between $y = 0$ and $y = 1/\alpha$.

3. Bounds for steady waves

Several of the bounds on wave properties obtained by KN, §3, will first be reconstructed, by arguments differing marginally from theirs. Then a new bound will be established which is crucial in confirming the BL conjecture. A variable with central importance in the present account is the flow velocity along the bottom of the channel, to be denoted by $U(x) := u(x, 0)$ together with $\check{U} = U(0)$ and $\hat{U} = U(\frac{1}{2}\lambda)$.

PROPOSITION 1. *As introduced in §2, let* $\hat{q}$ *and* $\check{q}$ *denote the flow velocity along the free surface at the wave crests and troughs, where respectively* $h = \hat{h}$ *and* $h = \check{h}$. *Then*

$$\hat{q}\hat{h} < 1 < \check{q}\check{h} \tag{3.1}$$

and

$$\hat{U}\hat{h} > 1 > \check{U}\check{h}. \tag{3.2}$$

Proof. At the troughs, property (iii) listed in §2 means that $\psi_{yy}(0,y) > 0$ for $y < 0 \leqslant \check{h}$. Hence, because $\psi(0,0) = 0$, $\psi(0,\check{h}) = 1$, $\psi_y(0,0) = \check{U}$ and $\psi_y(0,\check{h}) = \check{q}$, we have

$$0 < \int_0^{\check{h}} \psi_{yy}(0,y)y\,\mathrm{d}y = \check{q}\check{h} - \int_0^{\check{h}} \psi_y(0,y)\mathrm{d}y$$
$$= \check{q}\check{h} - 1$$

and

$$0 < \int_0^{\check{h}} \psi_{yy}(0,y)(\check{h} - y)\mathrm{d}y = -\check{U}\check{h} + 1,$$

which verify the second parts of (3.1) and (3.2). Similarly, the first parts of (3.1) and (3.2) follow from the property $\psi_{yy}(\frac{1}{2}\lambda, y) < 0$ for $0 < y \leqslant \hat{h}$ at the crests. □

The inequalities (3.1) recover (3.1) in KN, but (3.2) were not used there and will be particularly helpful at present. The next result recovers Proposition 1R in KN, p. 666.

PROPOSITION 2. *As defined by (2.3), the total-head constant* r *for a periodic wave-train satisfies* $r > 1$. *If* h_1 *and* h_2 *are the depths of the subcritical and supercritical uniform streams that have a prescribed* $r > 1$, *then*

$$h_2 < \check{h} < h_1 < \hat{h}. \tag{3.3}$$

Proof. Recalling (1.5) or (2.3), we have that the possible depths h of uniform streams, for which $\psi = y/h$, are positive roots of

$$r = \mathscr{R}(h), \tag{3.4}$$

where

$$\mathscr{R}(h) = \tfrac{1}{3}(2h + h^{-2}). \tag{3.5}$$

The graph of $\mathscr{R}(h)$ is shown in figure 5. Note that $\mathscr{R}(h)$ has an absolute minimum $\mathscr{R}(1) = 1$ for $h > 0$, and that $h_2 < 1 < h_1$ when $r > 1$.

According to (3.4) and (2.3), the inequalities (3.1) imply that

$$\tfrac{3}{2}\mathscr{R}(\hat{h}) = \hat{h} + \tfrac{1}{2}\hat{h}^{-2} > \hat{h} + \tfrac{1}{2}\hat{q}^2 = \tfrac{3}{2}r$$
$$= \check{h} + \tfrac{1}{2}\check{q}^2 > \check{h} + \tfrac{1}{2}\check{h}^{-2} = \tfrac{3}{2}\mathscr{R}(\check{h}). \tag{3.6}$$

Hence $\mathscr{R}(\check{h}) \geqslant 1$ implies $r > 1$.

Next, because $r = \mathscr{R}(h_1) = \mathscr{R}(h_2)$ by definition of h_1 and h_2, the inequality $r > \mathscr{R}(\check{h})$ included in (3.6) requires $h_2 < \check{h} < h_1$, as is evident from figure 5. Finally, because $\hat{h} > \check{h}$, the inequality $\mathscr{R}(\hat{h}) > r$ in (3.6) requires $\hat{h} > h_1$, which completes the demonstration of (3.3). □

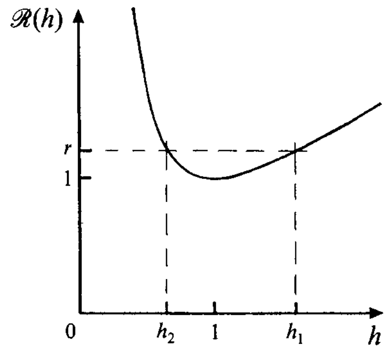

FIGURE 5. Graph of $\mathscr{R}(h)$ defined by (3.5).

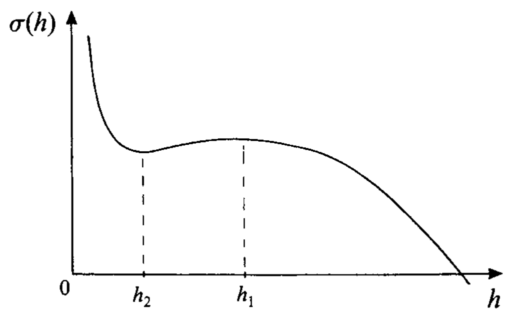

FIGURE 6. Graph of $\sigma(h)$ defined by (3.7), with $r > 1$.

The next result, which recovers Proposition 2 in KN, p. 668, refers to the function σ defined by

$$\tfrac{3}{2}\sigma(h) = \tfrac{3}{2}rh - \tfrac{1}{2}h^2 + \tfrac{1}{2}h^{-1}. \tag{3.7}$$

According to (2.4) or to (1.5), $\sigma(h)$ is the flow-force constant s for uniform streams with a prescribed $r > 1$; and (3.4) implies that $\sigma'(h_1) = 0$ and $\sigma'(h_2) = 0$. The graph of $\sigma(h)$ for $h > 0$ is shown in figure 6, which makes plain that $\sigma(h_2) < \sigma(h_1)$ when $r > 1$.

PROPOSITION 3. *Let $s_2 = \sigma(h_2)$ denote the value of s for the supercritical uniform stream with given $r > 1$. Then, for periodic waves with this r,*

$$s > s_2. \tag{3.8}$$

Proof. Evaluate the expression (2.4) for s at the wave troughs, where $\psi_x(0, y) = 0$ for all $y \in [0, \check{h}]$, and note that

$$\int_0^{\check{h}} \psi_y^2(0, y)\, \mathrm{d}y = \int_0^{\check{h}} \left[\left(\frac{1}{\check{h}}\right)^2 + \left\{\psi_y(0, y) - \frac{1}{\check{h}}\right\}^2\right] \mathrm{d}y$$

$$> \int_0^{\check{h}} \left(\frac{1}{\check{h}}\right)^2 \mathrm{d}y = \frac{1}{\check{h}}$$

in consequence of

$$\int_0^{\check{h}} \psi_y(0, y)\, \mathrm{d}y = 1.$$

Thus (2.4) shows that

$$s > \sigma(\check{h}),$$

whence (3.8) follows from (3.3), specifically from $h_2 < \check{h} < h_1$, and from the form of σ illustrated in figure 6. □

The crucial new bound can now be established as follows.

PROPOSITION 4. *Let $s_1 = \sigma(h_1)$ denote the value of s for the subcritical uniform stream with given $r > 1$. Then, for periodic waves with this r,*

$$s < s_1. \tag{3.9}$$

Proof. Because $U(x)$ and $h(x)$ are continuous functions, respectively monotonic decreasing and monotonic increasing on $[0, \frac{1}{2}\lambda]$, the inequalities (3.2) imply the existence of a number $\bar{x} \in (0, \frac{1}{2}\lambda)$ such that

$$\bar{U}\bar{h} = 1, \tag{3.10}$$

where $\bar{U} = U(\bar{x})$ and $\bar{h} = h(\bar{x})$. Consider the harmonic function $\psi - \bar{U}y$, which is zero along the bottom $y = 0$. Because $\check{U} > \bar{U} > \hat{U}$, this function is positive for all $y \in (0, \check{h}]$ at a wave trough and negative for all $y \in (0, \hat{h}]$ at a crest. Consequently, a path Γ on which $\psi - \bar{U}y = 0$ exists traversing D from $(\bar{x}, 0)$ to the free surface. Let us provisionally take Γ to be described by $x = \bar{x} + \xi(y)$ with $\xi(0) = 0$: that is,

$$\psi(\bar{x} + \xi(y), y) = \bar{U}y \qquad \text{on } \Gamma. \tag{3.11}$$

Supposing this path to reach the free surface $\psi = 1$ at $y = h$, we infer from (3.11) that $\bar{U}h = 1$ and so $h = \bar{h}$ because h is a monotonic function on $[0, \frac{1}{2}\lambda]$. Therefore $\xi(\bar{h}) = 0$. Thus the path Γ begins and ends at $x = \bar{x}$, although presumably deviating from $x = \bar{x}$ for $0 < y < \bar{h}$.

Differentiation of (3.11) gives

$$\psi_y + \psi_x \xi'(y) = \bar{U} \qquad \text{on } \Gamma, \tag{3.12}$$

which shows $|\xi'(y)|$ to be bounded on $[0, \bar{h}]$ because $\psi_x > 0$ for $0 < x < \frac{1}{2}\lambda$ and $y > 0$. Thus the function $\xi(y)$ is confirmed to be single-valued, and a parametric representation of Γ is unnecessary. Note also that, on $y = 0$, we have $\psi_{yy} = -\psi_{xx} = v_x = 0$, $\psi_{xy} = -v_y = U_x$ and $\psi_{xyy} = -v_{yy} = v_{xx} = 0$. Hence, as $y \downarrow 0$ along Γ,

$$\psi_y(\bar{x} + \xi(y), y) = \bar{U} + U_x(\bar{x}, 0)\xi + O(y^2),$$
$$\psi_x(\bar{x} + \xi(y), y) = U_x(\bar{x}, 0)y + O(y^3).$$

It follows from (3.12) that $\xi'(0) = 0$.

Now, because $\Delta\psi = 0$ in D, the line integral

$$\oint \{(\psi_y^2 - \psi_x^2)\mathrm{d}y + 2\psi_x\psi_y \mathrm{d}x\}$$

is zero around any closed circuit in D. The expression (2.4) for s can therefore be recast in the version

$$\tfrac{3}{2}s = \tfrac{3}{2}r\bar{h} - \tfrac{1}{2}\bar{h}^2 + \int_\Gamma \{\tfrac{1}{2}(\psi_y^2 - \psi_x^2)\mathrm{d}y + \psi_x\psi_y\mathrm{d}x\}, \tag{3.13}$$

which the aforementioned principle implies to be equivalent to the original version of (2.4) with the integral over the cross-section at $x = \bar{x}$. As $\mathrm{d}x = \xi'(y)\mathrm{d}y$ along Γ, the integral in (3.13) is

$$\int_\Gamma \tfrac{1}{2}\left\{\psi_y^2 - \psi_x^2 + 2\psi_x\psi_y\xi'(y)\right\}\mathrm{d}y,$$

which upon substitution for $\psi_x\xi'(y)$ from (3.12) becomes

$$\int_\Gamma (\bar{U}\psi_y - \tfrac{1}{2}\psi_y^2 - \tfrac{1}{2}\psi_x^2)\mathrm{d}y = \int_\Gamma \tfrac{1}{2}\{\bar{U}^2 - (\bar{U} - \psi_y)^2 - \psi_x^2\}\mathrm{d}y$$
$$< \int_0^{\bar{h}} \tfrac{1}{2}\bar{U}^2\mathrm{d}y = \tfrac{1}{2}\bar{U}^2\bar{h} = \tfrac{1}{2}\bar{h}^{-1}. \tag{3.14}$$

Here (3.10) is used finally.

Equation (3.13) and the inequality (3.14) establish that $s < \sigma(\bar{h})$, where σ is defined by (3.7). Hence, in view of $h_2 < \check{h} < \bar{h}$, the form of σ illustrated in figure 6 implies that $s < \sigma(h_1)$ as required. □

It has been proved that all steady periodic waves in a uniform horizontal channel have the property $s_2 < s < s_1$ for a given $r > 1$; and thus the BL conjecture is confirmed. Needless to say, this conclusion covers the corresponding property that $r_1 < r < r_2$ for a given $s > 1$, where r_1 and r_2 are the respective values of r for the subcritical and supercritical uniform streams with this s. Either property is tantamount to the statement that all possible values of r and s characterizing steady periodic waves lie inside the cusp-bounded region of the (r, s)-plane shown in figure 1. Equivalently, in terms of the discriminant

$$\Delta := 3r^2s^2 - 4(r^3 + s^3) + 6rs - 1$$

of the cubic $C(h)$ defined by (1.3), the present conclusion amounts to $\Delta > 0$ for all steady periodic waves, whereas $\Delta = 0$ for all uniform streams.

It remains to mention solitary-wave solutions of the water-wave problem, which are most conveniently treated with the origin of x relocated below the wave crest. Modelled as a steady motion, any solitary wave has by definition the same r and s as the uniform stream to which the flow is asymptotic as $x \to \pm\infty$; and this uniform stream is known to be always supercritical (Keady & Pritchard 1974, p. 348, Proposition 1; Amick & Toland 1981*a*; McLeod 1989). Thus the class of solitary waves with all possible amplitudes is represented by points (r, s) on the lower branch of the cusped curve in figure 1.

4. Waves on water of infinite depth

Uniform streams of large but finite depth correspond to points far along the upper branch of the cusped curve in figure 1, and periodic Stokes waves on deep water are represented within a narrow neighbourhood below this branch in the (r, s)-plane. According to their original definitions, however, Q, R and S are all unbounded in the limit of infinite depth, and so the properties of steady waves on deep water deserve an alternative representation as follows.

Take axes (x, y) with origin in the free surface, x horizontal and y upwards. (Here these variables and the dependent variables will not be made non-dimensional.) For steady waves, such that the velocity in the water is $(u, v) = (c, 0)$ at $y = -\infty$, the equation of the free surface is

$$y = \eta(x), \tag{4.1}$$

where η is a periodic real function depending on the velocity parameter $c > 0$ and having zero mean value. The steady wave motion is represented by a stream function $\Psi(x, y)$ such that $u = c + \Psi_y$ and so satisfying the kinematic free-surface condition

$$c\eta(x) + \Psi(x, \eta(x)) = 0 \qquad \forall\, x \in \boldsymbol{R}, \tag{4.2}$$

together with

$$\Delta\Psi = 0 \qquad \text{in} \qquad \boldsymbol{R} \times (-\infty, \eta(x)) \tag{4.3}$$

and

$$|\nabla\Psi| \to 0 \qquad \text{as} \qquad y \to -\infty. \tag{4.4}$$

(Note that equivalently $\Psi(x + ct, y)$ is the stream function for waves propagating with velocity c in the $-x$-direction on water that is stationary at infinite depth.) The dynamical free-surface condition (Bernoulli's equation) is

$$g\eta + \tfrac{1}{2}[(c + \Psi_y)^2 + \Psi_x^2] = \tfrac{1}{2}c^2. \tag{4.5}$$

Equations (4.2)–(4.5) are satisfied trivially by null functions for η and Ψ, corresponding to which the unperturbed free surface is the x-axis.

It must be emphasized that the asymptotic condition (4.4) does not require Ψ to vanish in the limit $y \to -\infty$. In fact, as is well known from a slightly different standpoint, we have

$$\lim_{y \to -\infty} \Psi(x, y) = A > 0 \qquad \forall\, x \in \boldsymbol{R}, \tag{4.6}$$

where the positive constant A is just the net volume flux induced by a progressive wave-train on water that is at rest in the limit $y \to -\infty$ (i.e. A is the drift originally estimated by Stokes 1847).

The flow-force invariant for steady waves on water of infinite depth was first noted by Benjamin (1984, p. 45, equation (6.9): see also Baesens & MacKay 1992). It is given by

$$\mathscr{S} = -\tfrac{1}{2}g\eta^2 + \int_{-\infty}^{\eta} \tfrac{1}{2}(\Psi_y^2 - \Psi_x^2)\mathrm{d}y, \tag{4.7}$$

from which one can easily confirm that $\mathrm{d}\mathscr{S}/\mathrm{d}x = 0$ when η and Ψ satisfy (4.2)–(4.5). Reconsidering the problem of steady waves on water of finite depth, and writing $H = H_1 + \eta$, $\psi = c(y + H_1) + \Psi$, one may also confirm that

$$\mathscr{S} = \lim_{H_1 \to \infty} (S - S_1),$$

where S is the flow force for waves with total head $R = gH_1 + \tfrac{1}{2}c^2$ and S_1 is its value for the corresponding uniform (subcritical) stream.

For the case of small-amplitude waves, (4.2)–(4.5) are satisfied by

$$\eta = \varepsilon \cos kx, \qquad \Psi = -c\varepsilon \mathrm{e}^{|k|y} \cos kx,$$

$$c^2 = g/|k|,$$

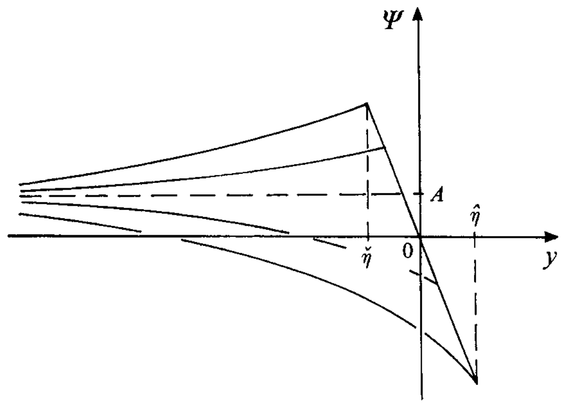

FIGURE 7. Graphs of Φ versus y for various values of x over half a wavelength of steady wave-train on deep water.

where ε is infinitesimal and $k \in \boldsymbol{R} \setminus \{0\}$. Hence, when approximated to $O(\varepsilon^2)$, (4.7) recovers the well-known result

$$\mathscr{S} = -\tfrac{1}{4}g\varepsilon^2$$

(Lamb 1932, §249), regarding which it should be recalled that $-\mathscr{S} > 0$ is wave resistance as usually defined. For infinitesimal waves, moreover, the classic estimate of Stokes drift is $A = \frac{1}{2}k\varepsilon^2 c$ (cf. Lamb 1932, p. 419, equation (16)).

Periodic waves on infinitely deep water, irrespective of wave amplitude, are known to have respective versions of the properties listed at the end of §2, and these basic properties will be taken for granted. In particular, writing as before $x = 0$ at a wave trough and $x = \frac{1}{2}\lambda$ at the next crest, we have that $\eta(x)$ is a monotonic increasing function of $x \in [0, \frac{1}{2}\lambda]$ from $\eta(0) = \check{\eta} < 0$ to $\eta(\frac{1}{2}\lambda) = \hat{\eta} > 0$. We also have that $u = c + \Psi_y$ is a monotonic decreasing function of $x \in [0, \frac{1}{2}\lambda]$ at all depths, and that $v = -\Psi_x > 0$ at all depths for $0 < x < \frac{1}{2}\lambda$. Furthermore, $u_y(0, y) = \Psi_{yy}(0, y) > 0$ for $y \in (-\infty, \check{\eta}]$, and $u_y(\frac{1}{2}\lambda, y) < 0$ for $y \in (-\infty, \hat{\eta}]$.

To illustrate these properties, graphs of $\Psi(x, y)$ against y at various stations x between 0 and $\frac{1}{2}\lambda$ are sketched in figure 7. The figure makes plain that, in the (y, Ψ)-plane, the straight line $\Psi \equiv A$ for $y \in (-\infty, 0]$ passes through the *interior* of the field $\mathscr{F}$ of curves Ψ versus y parameterized by $x \in [0, \frac{1}{2}\lambda]$, all of which are asymptotic to $\Psi \equiv A$ as $y \to -\infty$. Specifically, the graph of $\Psi(0, y)$ curves upwards above $\Psi \equiv A$ for $-\infty < y \leqslant \check{\eta}$, and the graph of $\Psi(\frac{1}{2}\lambda, y)$ curves downwards below $\Psi \equiv A$ for $-\infty < y \leqslant \hat{\eta}$. Moreover, because $\Psi_x < 0$ everywhere inside the region bounded by the curves $\Psi = \Psi(0, y)$, $\Psi = \Psi(\frac{1}{2}\lambda, y)$ and the straight line $\Psi = -cy$ representing the free surface, the line $\Psi \equiv A$ is covered injectively by the field $\mathscr{F}$.

Our principal deduction about the present form of the water-wave problem can now be stated.

PROPOSITION 5. *For all periodic waves on water of infinite depth, irrespective of amplitude $\hat{\eta} - \check{\eta} > 0$, the flow-force invariant $\mathscr{S}$ defined by (4.7) satisfies*

$$\mathscr{S} < 0. \tag{4.8}$$

Proof. The properties listed above imply that there is a station $\bar{x} \in (0, \frac{1}{2}\lambda)$ such that $\Psi(\bar{x}, \eta(\bar{x})) = A > 0$, and consequently $\bar{\eta} := \eta(\bar{x}) = -A/c \in (\check{\eta}, \hat{\eta})$ according to

(4.2). This $\bar{x}$ is unique because η is a monotonic increasing function over the interval $(0, \frac{1}{2}\lambda)$. Evaluating (4.7) at $x = \bar{x}$, one has

$$\mathscr{S} = -\tfrac{1}{2}g\bar{\eta}^2 + \int_{-\infty}^{\bar{\eta}} \tfrac{1}{2}\{\Psi_y^2(\bar{x}, y) - \Psi_x^2(\bar{x}, y)\}\mathrm{d}y. \tag{4.9}$$

For the same reason as used to derive (3.13) from (2.4), the integral in (4.9) is equivalent to a line integral along the path Γ described by $x = \bar{x} + \xi(y)$, where ξ is determined by satisfying $\xi(\bar{\eta}) = 0$, $\xi(-\infty) = 0$ and

$$\Psi(\bar{x} + \xi(y), y) = A \quad \text{for} \quad -\infty < y \leqslant \bar{\eta}. \tag{4.10}$$

Thus we infer

$$\mathscr{S} = -\tfrac{1}{2}g\bar{\eta}^2 + \int_\Gamma \{\tfrac{1}{2}(\Psi_y^2 - \Psi_x^2)\mathrm{d}y + \Psi_x\Psi_y\mathrm{d}x\}. \tag{4.11}$$

The properties affirmed above with reference to figure 7 ensure that the path Γ is well defined, for it is equivalent to the straight line $\Psi \equiv A$ in the (y, Ψ)-plane. In particular, differentiation of (4.10) shows that

$$\Psi_y + \Psi_x\xi'(y) = 0 \quad \text{on}\, \Gamma, \tag{4.12}$$

implying that $\xi : (-\infty, \bar{\eta}) \to \boldsymbol{R}$ is a single-valued function because $\Psi_x < 0$ everywhere on Γ.

By use of $\mathrm{d}x = \xi'(y)\mathrm{d}y$ and then (4.12), it follows immediately from (4.11) that

$$\mathscr{S} = -\tfrac{1}{2}g\bar{\eta}^2 - \int_\Gamma \tfrac{1}{2}(\Psi_x^2 + \Psi_y^2)\mathrm{d}y < 0,$$

and thus (4.8) is confirmed. □

For periodic waves on infinitely deep water the inequality (4.8) is, of course, the counterpart of (3.9) for waves on water of finite depth. It too appears to be proven here for the first time.

5. Discussion

5.1. *Amplitude dependence*

The BL conjecture has been vindicated without need to examine how wave amplitude and wavelength depend on r and s within the admissible region of the (r, s)-plane. However, several facts about this aspect are worth recalling here.

For long steady waves of small amplitude, represented by very small values of $r - 1 > 0$ and $s - 1 > 0$, the approximation (1.2) becomes reliable and according to it the range of possibilities is easily inferable, as has already been summarized in §1. Namely, for fixed $r > 1$, wave amplitude and wavelength increase monotonically as s is reduced from s_1 to $s_2 < s_1$. Periodic waves become infinitesimal as $s \uparrow s_1$, and in this limit their wavelength $\lambda = 2\pi/|k|$ is determined by the well-known dispersion relation $F_1^2 = (\tanh|k|\,h_1)/|k|\,h_1 = 1 - \frac{1}{3}k^2h_1^2 + O(k^4h^4)$ with $F_1^2 = h_1^{-3} < 1$. In the other limit $s \downarrow s_2$, one has $\lambda \to \infty$ and the solitary-wave solution of (1.3) has amplitude $\hat{h} - h_2 = h_2^{-2} - h_2 = h_2(F_2^2 - 1)$, which is the maximum possible for the given $r > 1$.

In the light of the general results established in §3, it can be inferred that this ordering of the possibilities remains qualitatively unchanged until r is raised to values marginally less than its value for the solitary wave of extreme form (i.e. $r = 1.031$ approximately). Schwartz (1974), Longuet-Higgins (1975), Cokelet (1977) and others have shown, however, that gross properties of steady waves, such as r and s, have turning points in respect of dependence on amplitude a little way short of reaching waves of extreme form (whose crests are stagnation points).

It was suggested by BL that the (r, s)-chart of possible steady waves might be completed by a third barrier representing waves of extreme form. But, for the reason just noted which is a more recent finding, realizable points (r, s) in fact lie on either side of this locus. The (r, s)-chart of possibilities nevertheless includes such a barrier, shown as a dashed line in figure 1, which is close to the locus of extreme, sharp-crested waves and to the right of which no steady wave exists.

5.2. *Bifurcated steady waves*

It has been known for the last 15 years that the problem of steady water waves has solutions other than Stokes waves. Subsequent to the original discoveries reported by Chen & Saffman (1980) and Saffman (1980) as regards waves on deep water, many other authors have contributed to this aspect of the steady-wave problem (e.g. Vanden-Broeck 1983; Longuet-Higgins 1985; Zufiria 1978*a,b*; Baesens & MacKay 1992). The phenomena in question arise upon waves of large amplitude, in the range where the class of Stokes waves becomes parametrically multivalued: more specifically, where the relation between amplitude and two other parameters – such as (r, s) for waves on water of finite depth – undergoes turning points.

These phenomena are best understood by treating the steady-wave problem somehow as an evolutionary system with the horizontal coordinate x in the role of time (cf. Kirchgässner 1988). From this viewpoint, the simple representation of possible solutions $h(x)$ illustrated in figure 2 with reference to long waves of small amplitude can be reckoned to remain qualitatively the same for waves of moderate amplitude, but to be replaced by more complicated behaviour when a fold appears in the solution set. Then period-doubling can occur, which breaks the symmetry of Stokes waves about both their crests and troughs, and which is representable in the standard way as a flip bifurcation of the Poincaré return map for one Stokes-wave period (Guckenheimer & Holmes 1983, §3.2). Period-tripling and various other, more intricate possibilities also appear to arise, the most thorough account of which so far has been given by Baesens & MacKay (1992) for steady waves on deep water. Needless to say, these symmetry-breaking bifurcations from the Stokes class of steady waves are closely linked with time-dependent stability properties of the steady waves in question; but no comment on this aspect will be made here.

The device of formulating the steady two-dimensional water-wave problem as a dynamical system in respect of x-dependence is admittedly artificial, because the underlying field equation – Laplace's equation – is elliptic and so any such evolutionary system is ill-posed on any open set of initial data. The formulation can well be rationalized, however, by a presumed restriction of its scope to a finite *centre manifold* of solutions that are bounded on $\boldsymbol{R}$, including Stokes waves whose existence is provable otherwise and also including uniformly bounded perturbations of this basic class of periodic solutions (for a full discussion, see Kirchgässner 1988, Ch. VI). Indeed, this rationale is imperative to a neat theoretical explanation of symmetry-breaking bifurcations from Stokes waves, which were first detected numerically.

Representations of the steady water-wave problem in Hamiltonian form have been found by Mielke (1991) for the case of finite depth and by Baesens & MacKay (1992) for the case of infinite depth, being particularly useful in accounting for the bifurcations of steady waves. They exemplify a general recipe that was noted by Benjamin (1984, §4.4) to give a Hamiltonian structure to steady-wave problems governed by elliptic equations. The flow-force invariant $\frac{3}{2}s$ in the case of finite depth (or $\mathscr{S}$ for infinite depth) is always the Hamiltonian function in such representations, a few details of which will be summarized in Appendix B.

5.3. *Values of (r, s) for bifurcated steady waves*

Finally, to supplement the foregoing vindication of the BL conjecture, it will be argued that all possible subharmonic bifurcations from Stokes waves correspond to folds, perhaps multiple ones, *within* the cusp-bounded region of the (r, s)-chart shown in figure 1. That is, any such steady wave has $s_2 < s < s_1$ for its respective $r > 1$.

At first sight the inequality $s_2 < s$ is the more suspect in this connection, because there exist long periodic waves of large amplitude with (r, s) arbitrarily close to, although distinct from, the lower branch of the cusp in figure 1. Such waves are virtually a periodic succession of solitary waves near to extreme form, and they appear to be subject to symmetry-breaking bifurcations (Zufiria 1987*a*).

A revision of the proof of Proposition 3 in §3 covers all such possibilities. At a wave trough, say at $x = 0$ as in §3, we still have $h_x(0) = 0$ and therefore $\psi_x(0, h(0)) = 0$ by (2.2). As the Z_2 symmetry of the motion relative to the troughs may be broken, however, it can no longer be claimed that $\psi_x(0, y) = 0$ for $0 \leqslant y < h(0) = \check{h}$. A transverse path Γ can nevertheless be found, described by $x = \xi(y)$ for $y \in [0, \check{h}]$ with $\xi(\check{h}) = 0$, such that

$$\psi_x = 0 \quad \text{and} \quad \psi_{xx} \leqslant 0 \qquad \text{on } \Gamma,$$

i.e.

$$\psi_x(\xi(y), y) = 0 \quad \text{and} \quad \psi_{xx}(\xi(y), y) \leqslant 0 \quad \forall\, y \in [0, \check{h}]. \tag{5.1}$$

Thus Γ is composed of points at which streamlines $\psi = \text{const.} \in (0, 1]$ have minimum heights. It is evident, moreover, that

$$\psi_{xx}(\xi(y), y) < 0 \qquad \text{a.e. on } (0, \check{h}], \tag{5.2}$$

because otherwise (with equality in the second of (5.1) over any part of $(0, \check{h}]$ having finite measure) the harmonic function ψ would have an interior plateau, which is impossible.

The definition (5.1) of Γ implies that

$$\psi_{xx}\xi'(y) + \psi_{xy} = 0 \qquad \text{on } \Gamma. \tag{5.3}$$

Also, if $f(y) := \psi(\xi(y), y)$, then

$$f'(y) = \psi_x\xi'(y) + \psi_y = \psi_y \qquad \text{on } \Gamma,$$

and by (5.1) together with (5.3)

$$\begin{aligned} f''(y) &= \psi_x\xi''(y) + 2\psi_{xy}\xi'(y) + \psi_{yy} + \psi_{xx}[\xi'(y)]^2 \\ &= -\{[\xi'(y)]^2 + 1\}\psi_{xx} \qquad \text{on } \Gamma \\ &> 0 \qquad \text{a.e. for} \qquad 0 < y < \check{h}. \end{aligned} \tag{5.4}$$

Because $f(0) = 0, f(\check{h}) = \psi(0, \check{h}) = 1$ and $f'(\check{h}) = \psi_y(0, \check{h}) = \check{q}$, we infer from (5.4)

$$0 < \int_0^{\check{h}} f''(y) y \,\mathrm{d}y = \check{q}\check{h} - 1, \tag{5.5}$$

which generalizes the first of the inequalities (3.1).

By use of (5.5) in the definition (2.3) of r compared with (3.5), it follows at once that in the present case too

$$h_2 < \check{h} < h_1, \tag{5.6}$$

which generalizes the first of the inequalities in (3.3).

Next, by the reasoning already exemplified in §3, the definition (2.4) of s can be recast in its equivalent form

$$\begin{aligned}\tfrac{3}{2}s &= \tfrac{3}{2}r\check{h} - \tfrac{1}{2}\check{h}^2 + \int_\Gamma \{\tfrac{1}{2}(\psi_y^2 - \psi_x^2)\mathrm{d}y + \psi_x\psi_y\mathrm{d}x\} \\ &= \tfrac{3}{2}r\check{h} - \tfrac{1}{2}\check{h}^2 + \int_0^{\check{h}} \tfrac{1}{2}[f'(y)]^2\mathrm{d}y.\end{aligned}$$

(Note that an integral along the bottom $y = 0$ completes the equivalence with the integral over the cross-section at $x = 0$ as in (2.4), but the integral along the bottom is obviously zero.) In view of (5.4), the boundary conditions on f and then (5.5), the preceding expression implies that

$$\tfrac{3}{2}s > \tfrac{3}{2}r\check{h} - \tfrac{1}{2}\check{h}^2 + \tfrac{1}{2}\check{h}^{-1} = \tfrac{3}{2}\sigma(\check{h}). \tag{5.7}$$

Hence (5.6) and the form of σ illustrated in figure 6 establish that $s > s_2$, as required.

As regards the complementary inequality $s < s_1$ for bifurcated steady waves, an adaption of the proof of Proposition 4 in §4 is straightforward. An outline will suffice here. First, the reasoning used to establish the inequality $\check{U}\check{h} < 1$ in (3.2), now referred to the function $f''(y)$ which is positive on $(0, \check{h}]$ according to (5.4), shows that $U(x)$ still takes a value $\check{U} < 1/\check{h}$ somewhere under a wave trough – although now perhaps not exactly under. Moreover, by the definition of the path Γ, the value $\check{U}$ is still the maximum of $U(x)$, so that $\check{U} \geqslant U(0) > 0$. In consideration of a transverse path along which $\psi_x = 0$ under a wave crest, a similar argument shows that $U(x)$ still takes a minimum value $\hat{U} > 1/\hat{h}$ somewhere under the crest. It follows that a number $\bar{x}$ must still exist such that $U(\bar{x})h(\bar{x}) = 1$, and hence Proposition 4 can be proven in the same way as before. We can thus conclude that no bifurcated steady wave-train exists beyond the scope of the BL conjecture.

I am indebted to Dr G. Keady and to Professor J.F. Toland for constructive comments on the first draft of this paper.

Appendix A. Deduction of properties of Stokes waves

We start from the basic properties illustrated in figure 3, namely (*a*) h and ψ are periodic in x with period λ, (*b*) the horizontal bottom $y = 0$ is the streamline $\psi = 0$, (*c*) the free surface $y = h(x)$ is the streamline $\psi = 1$, (*d*) $\Delta\psi = 0$ in the flow domain D, (*e*) the vertical velocity $-\psi_x = 0$ for $x = 0$ and $x = \frac{1}{2}\lambda$, and (*f*) the vertical velocity $-\psi_x > 0$ for all $(x, y) \in (0, \frac{1}{2}\lambda) \times (0, h(x)] = D'$, say. Further properties including those listed at the end of §2 are deducible as follows by the maximum principle and Hopf's boundary-point lemma for harmonic functions (Gilbarg & Trudinger 1977, p. 33, Lemma 3.4).

1. Being harmonic, ψ takes its maximum and minimum values on the boundary of D', at points where the outward normal derivative of ψ is respectively positive and negative. Because $\psi_x = 0$ for $x = 0, \frac{1}{2}\lambda$, the maximum and minimum are at the free surface and the bottom respectively. Hence $0 < \psi < 1$ in the interior of D'. Moreover,

because the maximum $\psi = 1$ is attained at every point of the free surface, it follows by Hopf's lemma that $\psi_y > 0$ everywhere on the free surface, except in the case of waves of extreme form when the crest is a stagnation point. Similarly, $\psi_y > 0$ on the bottom, where the minimum $\psi = 0$ is attained at every point.

2. Being harmonic, ψ_y takes its minimum value on the boundary of D', at a point where its outward normal derivative is negative. Hence, because $\psi_{yx} = 0$ for $x = 0, \frac{1}{2}\lambda$, and $\psi_y \geqslant 0$ on the other boundaries of D', it follows that $\psi_y > 0$ in the interior of D'. Thus the horizontal velocity component $u = \psi_y$ is shown to be positive everywhere in the flow domain, except at the crest in the case of waves of extreme form.

3. Because the harmonic function ψ_x is given to be negative in the interior of D', and $\psi_x = 0$ on the bottom and for $x = 0, \frac{1}{2}\lambda$, Hopf's boundary-point lemma implies that

$$\psi_{yy} = -\psi_{xx} > 0 \quad \text{for} \quad x = 0,$$
$$\psi_{yy} = -\psi_{xx} < 0 \quad \text{for} \quad x = \tfrac{1}{2}\lambda,$$

and

$$\psi_{xy} < 0 \quad \text{for} \quad y = 0,\ 0 < x < \tfrac{1}{2}\lambda.$$

These inequalities confirm (iii) and the first part of (v) in §2.

4. That every streamline $\psi = \text{const.} > 0$ rises monotonically from its lowest point at $x = 0$ to its highest point at $x = \frac{1}{2}\lambda$ follows at once from the inequalities $\psi_x < 0 < \psi_y$ in the interior of D' (cf. (iv) in §2).

5. Property (ii) in §2 follows from Hopf's lemma referred to the function $\phi_x^2 + \phi_y^2$, which is superharmonic in D' and whose normal derivative on the boundary of D' is non-zero only at the free surface.

Appendix B. Hamiltonian formulations

A rudimental version may be recognized from the first (infinitesimal) variation of the flow-force equation (2.4). Thus, writing $P := -\psi_x$, we get from (2.4)

$$\tfrac{3}{2}\dot{s} = \{\tfrac{3}{2}r - h + \tfrac{1}{2}(\psi_y^2 - P^2)_{y=h}\}\dot{h} + \int_0^1 (\psi_y\dot{\psi}_y - P\dot{P})\mathrm{d}y. \tag{B 1}$$

As $\psi(x,0) = 0$ and $\psi(x,h(x)) = 1\ \forall\ x \in \boldsymbol{R}$, these conditions require

$$\dot{\psi}(x,0) = 0, \qquad \dot{\psi}(x,h(x)) + \psi_y(x,h(x))\dot{h} = 0.$$

Hence an integration by parts and imposition of the dynamical condition (2.3) at the free surface reduce (B1) to

$$\tfrac{3}{2}\dot{s} = -\int_0^h (\psi_{yy}\dot{\psi} + P\dot{P})\mathrm{d}y. \tag{B 2}$$

Thus Laplace's equation for ψ in D has the formal Hamiltonian representation

$$\psi_x = -P = \frac{3}{2}\frac{\delta s}{\delta P}, \qquad P_x = -\psi_{xx} = \psi_{yy} = -\frac{3}{2}\frac{\delta s}{\delta \psi}. \tag{B 3}$$

Here the variational derivatives of $\frac{3}{2}s$ subject to the condition at $y = 0$ and the two conditions at $y = h(x)$ are inferrred from (B2) in the standard way. This version of the Hamiltonian formalism is somewhat equivocal, however, because the upper limit $h(x)$ of the integrals in (2.4) and (B2) is part of the overall solution.

When y is taken as the dependent variable and (x, ψ) as the independent variables, a rather more clear-cut Hamiltonian structure is obtained, although at the expense of a more complicated elliptic equation controlling the centre manifold. Writing $y = y(x,\psi)$, so that $y(x,0) = 0$ and $y(x,1) = h(x)$, we deduce that the velocity components are given by

$$u = \frac{1}{y_\psi}, \qquad v = \frac{y_x}{y_\psi} \tag{B 4}$$

(cf. Benjamin 1966, equations (3.2)), and this representation of them is well defined because we know that $y_\psi > 0$ everywhere in $\boldsymbol{R} \times [0,1]$. The theory of implicit functions also tells us that, for $y =$ const.,

$$\psi_{xx} = -\frac{1}{y_\psi^3}(y_\psi^2 y_{xx} - 2y_x y_\psi y_{x\psi} + y_x^2 y_{\psi\psi}),$$

and, for $x =$ const.,

$$\psi_{yy} = \frac{1}{y_\psi}\left(\frac{1}{y_\psi}\right)_\psi = \frac{1}{2}\left(\frac{1}{y_\psi^2}\right)_\psi .$$

Hence the condition of zero vorticity, $\Delta\psi = 0$, is equivalent to

$$\frac{y_{xx}}{y_\psi} - \frac{2y_x y_{x\psi}}{y_\psi^2} - \tfrac{1}{2}(1+y_x^2)\left(\frac{1}{y_\psi^2}\right)_\psi = 0 \tag{B 5}$$

(cf. Benjamin 1967, equation (3.3)).

In the new variables, (2.4) can be rewritten

$$\tfrac{3}{2}s = \tfrac{3}{2}rh - \tfrac{1}{2}h^2 + \int_0^1 \frac{1}{2}\left(\frac{1}{y_\psi} - \frac{y_x^2}{y_\psi}\right) \mathrm{d}\psi \tag{B 6}$$

$$= \tfrac{3}{2}rh - \tfrac{1}{2}h^2 + \int_0^1 \frac{1}{2}\left(\frac{1}{y_\psi} - y_\psi P^2\right) \mathrm{d}\psi, \tag{B 7}$$

where $P := -y_x/y_\psi$ is the first derivative of the integrand in (B6) with respect to y_x, so being the generalized-momentum variable for the present Hamiltonian formulation. Differentiating (B7) with respect to x, then integrating by parts and applying the dynamical boundary condition (2.3), which now takes the form

$$\tfrac{3}{2}r = h(x) + \frac{1}{2}\left\{\frac{1}{y_\psi^2(x,1)} + P^2(x,1)\right\}, \tag{B 8}$$

one can readily reconfirm from (B7) that $\mathrm{d}s/\mathrm{d}x = 0$ when $y(x,\psi)$ satisfies (B5), which is equivalent to

$$\frac{1}{2}\left(\frac{1}{y_\psi^2}\right)_\psi + PP_\psi + P_x = 0. \tag{B 9}$$

The first variation of (B7) is

$$\tfrac{3}{2}\dot{s} = \left(\tfrac{3}{2}r - h\right)\dot{h} - \int_0^1 \left(\frac{\dot{y}_\psi}{2y_\psi^2} + \tfrac{1}{2}\dot{y}_\psi P^2 + y_\psi P\dot{P}\right) \mathrm{d}\psi.$$

Hence, because $\dot{y}(x,1) = \dot{h}(x)$, an integration by parts leads to cancellation of the integrated terms by virtue of (B8) and the outcome is

$$\tfrac{3}{2}\dot{s} = \int_0^1 \left[\left\{\frac{1}{2}\left(\frac{1}{y_\psi^2}\right)_\psi + PP_\psi\right\}\dot{y} - y_\psi P\dot{P}\right] \mathrm{d}\psi. \tag{B 10}$$

The variational equation (B10) shows that

$$\left.\begin{aligned} \frac{3}{2}\frac{\delta s}{\delta P} &= -y_\psi P = y_x, \\ -\frac{3}{2}\frac{\delta s}{\delta y} &= -\frac{1}{2}\left(\frac{1}{y_\psi^2}\right)_\psi - PP_y = P_x \end{aligned}\right\} \tag{B 11}$$

if y satisfies (B9). Thus a canonical Hamiltonian structure is demonstrated for the x-dependent quasi-evolutionary problem in terms of the pair of variables $(y,P)(x,\psi)$, which are defined on the fixed domain $\boldsymbol{R}\times[0,1]$ and are required to satisfy $y = P = 0$ on $\psi = 0$ together with the Bernoulli equation (B8) on $\psi = 1$. Again $\frac{3}{2}s$ is the Hamiltonian functional, of (y,P) rather than of (ψ,P) as before. But the present Hamiltonian representation is evidently clearer than the previous one in that the integrals in (B7) and (B10) are over a fixed interval.

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